Pascal, Fermat, and the Birth of Probability Theory

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Northern Italy, 1494
Luca Pacioli
Luca Pacioli

1445-1517. Born to a family of “abbaci.” Career spent as a Franciscan friar and teacher of arithmetic in northern Italy. Taught mathematics to Leonardo da Vinci. His *Summa di Arithmetica et Geometria* is the ultimate exemplar of the abbaco tradition of teaching arithmetic and algebraic problem-solving (beginning with Leonardo’s *Liber Abaci* in 1202; about 250 manuscripts are now known).

(Algebra was not (routinely) taught in universities until the late 17th century; before then, it was part of the vernacular tradition of the “abbacus” schools along the western Mediterranean coast. Texts used numerous examples to focus on practical arithmetical problem solving.)
The Summa
Illustrations in the *Summa*
The Division Problem

Two teams compete in a game of skill, such as the *Jeu de la Longue Paume* (lawn tennis), intending to play until one first wins six rounds, with the winner to take the entire prize. How should the prize be divided if the game is interrupted with the score 5 to 2?

Pacioli’s solution: in the proportion 5:2.
Issues

• What is the origin of this class of questions? Why would they seem important?
• Must the game be of skill, purely of chance, or does it not matter?
• Are there any principles allowing the resolution of such questions?
• What are the solutions? How can they be extended to multiple players?
• What, if any, were the implications and applications of such solutions?
Trade in the Middle Ages

When the cargo is shared among merchants, which should be jettisoned? That is, how should damages be divided?

“General average” was the early principle (dating to Rhodes, 900 BCE): each interested party contributed proportionally to make good those who lost (compare to “particular average” falling on only one interest).

**Average**: “financial loss incurred through damage to goods in transit.” (French *avarie*, “damage to ship”; Italian *avaria*, c. 12th century.) Meaning shifted to “equal sharing of such loss by the interested parties.” The mathematical usage dates only to 1755.
Contracts in the XIII\textsuperscript{th} Century

A “command contract” concerns people who invest capital as well as labor for varying periods of time. If the enterprise ends prematurely, how should the assets be distributed?

“Two partners form a company in which one invests 300 pounds and the other 200. It is to last for 3 years. They agree that the profit at the end will be evenly divided. But at after 20 months, they wish to split up and find they have a 200 pound profit. How much should each get?” – Paolo Gherardi, \textit{Trattato di ragioni} (Florence), 1328.

Such problems were generally solved with \textit{proportions} based on the amounts of capital, labor, and passage of time involved.
An Origin Hypothesis

…The division problem is the invention of an arithmetic master familiar with the practices and modes of thought of merchants and gamblers, or a merchant who was at once a gambler and calculator; an invention … made possible by the modeling [and abstraction] already created by broken contracts on the one hand and games that take several rounds on the other hand. In these conditions it becomes possible … to ask what might happen, precisely because the situation is sufficiently simple to foresee several possible futures that are very easily identified. … This is entirely different from the situation encountered by those who had forgotten the origin of the problem and knew only its statement [i.e., from the end of the XVth century onwards]. – Norbert Meusnier.
Girolamo Cardano

1501-1576(?). Illegitimate child. Studied medicine but not admitted to the College in Milan. Practiced as a physician. (Effected a famous cure of the Archbishop of St. Andrews, 1553.)

The “gambling scholar.” Wrote about medicine, mathematics, gambling, physics, and much more.

Published general solutions to the cubic in *Ars Magna*, 1545, using negative and imaginary numbers.

*Liber de Ludo Alaea* (1526) includes probability—but was not published until 1663.
Girolamo Cardano (1539)

Consider a game to 19 with the score 18 to 9. Pacioli’s proportion divides the stakes $18:9 = 2:3$ even though the first player needs to win only 1 game and the second, 10 games. “Most absurd!” (Pratica arithmetica et mensurandi singularis, 1539.) Other absurd cases: (19, 2, 0) and (19, 18, 0).

Cardano’s approach

Suppose we start a game in which I have to win three and you one round only. How much should you have to wager if my bet is 1?
Cardano’s Solution

If we only had to play one round, the wager should be even and you would bet 1. [And so by winning I would have 2.] If we played two rounds, you should have to *triple* your bet: because in winning two games [to win the prize], I would then gain 4, after having run the risk first of losing my 1 and then of losing my 2. With one more round, the difficulty is redoubled: I now have 6 at risk. Thus *your bet should be sextuple mine*.

(Recounted by Ernest Coumet, *Le Probleme des Partis avant Pascal*, 1965. Coumet speculates the series would continue 10, 15, ….)
Cardano’s Principles

• Two players will be considered to have equal chances when the conditions under which they play are identical for both.

• The calculation of probabilities is an application of combinatorics.

[After Boutroux, Les Origines du Calcul des Probabilités, 1980.]
Niccolò Fontana (‘‘Tartaglia’’)

1500-1557. Self-taught Venetian mathematician, engineer, surveyor, and bookkeeper. Impoverished math teacher from the age of 16 onwards.

Translated Euclid and Archimedes. His *General Trattato di numeri, et misure* (1556) is one of the most acclaimed 16th century European arithmetic texts.

Won the 1535 contest with Fior (del Ferro’s student) to solve cubics.

Fell out with Cardano, who published Tartaglia’s secret formula for the cubic equation (1545).
Tartaglia’s Defeat (1556)

“la risoluzione di una tale questione è più presto giudiciale, che per ragione” (the solution of such a problem is a judicial one rather than one of reasoning).

Tartaglia recognized that the problem was one of determining the value of a position, but could not identify a (unique) solution.
“Ohri”

An Italian manuscript c 1380 found in a pile of anonymous undated manuscripts in a library in Siena. (Perhaps is the one referred to by Oystein Ore (1960).) studied by Norbert Meusnier in the 1990’s.

Two chess players stop after one is ahead 2-0 in a game to 3 rounds (with a prize of 2 ducats). The original idea is that each round contributes a separate and potentially different value to the winnings. At each turn, we consider how much is won or lost and explore all the possibilities.
Ohri’s Interrupted Chess Game (1380)

Key: $x$-$y$ means the first player has won $x$ games and the second has won $y$ games.
Ohri’s Calculation: Initialization

Key: $v$ is the current value of the game for the first player (whence $2-v$ is the value for the second player).
Ohri’s Calculation (Step 1)

Key: Edges are labeled with the first player’s gains.
Ohri’s Calculation (Step 2)

The loss for 1-0 → 1-1 is found by subtracting 1 from $1+c$. 
Symmetry (fairness) says the gain for 1-0 \rightarrow 2-0 is the negative of that for 1-0 \rightarrow 1-1. We now know the value of 2-0.
Ohri’s Calculation (Step 4)

The gain at 2-0 → 3-0 is found by subtracting the start value $1+2c$ from the final value 2.
Again, symmetry tells us the change for $2-0 \rightarrow 2-1$ must be $-(1-2c)$.
Ohri’s Calculation (Step 6)

The values at 2-1 are obtained via subtraction. Symmetry now tells us \(-(1-4c) = 2-4c\), whence \(c = 3/8\).
Ohri’s Result

The values for first player; second player are shown.
Pascal and Fermat
Pascal to Fermat, Wednesday July 29

Your method is quite certain, and is that which first came to my mind in this study; but, because the pain of combinations is excessive, I found a short-cut and indeed another method, much shorter and more elegant …

This is how I determine the value of each of the parts, when two players play, for example, for *three* wins, and each has put 32 pistoles into play:

Let us say that the first has won *two* and the other has won *one*…

\[
\begin{align*}
? & \quad 64 & = & \quad ? & \quad 64 & = & \quad 48 \\
0 & \quad 64 & \quad 32 & \quad 48+16 & \quad 48-16
\end{align*}
\]
Pascal’s Recursion (Step 2)

Now let us say that the first has won *two* rounds and the other has won *none*

\[
64 = 56 + 8
\]

\[
48 = 56 - 8
\]

(Previous situation)
Pascal’s Recursion (Step 3)

Finally, let us say that the first has won only \textit{one}, and the other \textit{none}.

\[
\begin{align*}
(\text{Previous situation}) & \quad 44+12 \\
32 & = \ldots = 44 \\
44-12 & 
\end{align*}
\]
Pascal’s Recursion: Summary

… Now by this means you see by simple subtractions, that in the first round, 12 pistoles were at stake; 12 in the second; and 8 for the last.
Pascal’s Formula for the Initial Value

Given as many rounds to be won as one would like, find the value of the first.

\[
\frac{1}{2} \binom{2n}{n} = \frac{(2n-1) \cdot (2n-3) \cdots \cdot 3 \cdot 1}{2 \cdot 4^{n-1}} = \frac{(2n) \cdot (2n-2) \cdots \cdot 4 \cdot 2}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16}
\]

E.g., the value of the first of eight [actually, nine] rounds is

\[
\frac{1}{2} \binom{18}{9} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16}
\]

which is to say that if each bets the number of pistoles expressed by the product of the [first eight] even numbers, he will get an amount of the other’s money expressed by the product of the [first eight] odd numbers. This can be demonstrated, but with much pain, by combinations such as you have imagined, and I was not able to demonstrate it by this other path that I have just told you, but only by means of combinations.
Pascal's General Formula

\[ f(k, n) = 256 \frac{2^{k-1}}{2^{2n}} \binom{2n - k - 1}{n - 1} \]

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Si on joue chacun 256, en

Il m'appar-tient sur les 256 pistoles de mon joueur, pour la
The "Arithmetical Triangle"
…the first thing that must be understood is that the money wagered by the players no longer belongs to them … they have received in return the right to wait for whatever chance might bring them, following the conditions they have agreed upon beforehand.

…they may stop voluntarily … and return to each one the property that is his due … which must be proportional to what they have the right to expect by chance. This is the *division* ("parti").
Principles

1. A gain that is certain to occur need not be shared.

2. If one person stands to win $x$ and the other $y < x$ and they have equal chances, the amount at risk will be split equally.

   **Corollary 1:** The split will be $y + (x - y)/2$.
   **Proof:** The amount $y$ is not at risk and $x - y$ is at risk.

   **Corollary 2:** The split will be $(x + y)/2$.

3. In general, all that matters is how many rounds need to be won, not how many have been played, because only the future rounds determine the current *state* of the game.
Pascal’s Published Solution

Let one player need \( k \) points and the other \( m \) points to win. In the \( k + m \) row of the triangle [nowadays indexed by \( k + m - 1 \)], take the sum of the first \( k \) values and the sum of the last \( m \) values: that is the proportion giving the relative advantages of the two players.

The proof is by induction on the rows. The first row is trivial: when both players need to win one point, the ratio must be \( 1/(1+1) \).

For the induction step, we need a Lemma: \( \left( \frac{a}{b} + \frac{c}{b} \right)/2 = \frac{a + c}{2b} \)
The Induction

Look at this canonical example using rows 4 and 5 of the triangle and suppose one player needs 2 points and the other 3.

\[
\begin{array}{c|c|c|c}
D & B & \theta & \lambda \\
H & E & C & R & \mu \\
3 & 2
\end{array}
\]

The claim is that the value to the first player is \((H+E+C)/(H+E+C+R+\mu)\). Inductively, a win puts him in a position worth \((D+B+\theta)/(D+B+\theta+\lambda)\) whereas a loss gives a position worth \((D+B)/(D+B+\theta+\lambda)\). By the lemma, this position’s value therefore equals \((2D+2B+\theta)/(2D+2B+2\theta+2\lambda)\) = \((H+E+C)/(H+E+C+R+\mu)\) (by a general property of the triangle), \textit{QED}. 
The Case of Three Players

Pascal to Fermat, August 24, 1654.

“When there are only two players, your method, which proceeds by combinations, is very certain; but when there are three, I believe I have a demonstration that it is not correct, although perhaps only because you proceed in a manner that I do not understand.”

Consider (3,2), which will be finished within 4 rounds, “whence you conclude that it is necessary to see how four rounds may be combined among two players, and see how many combinations make the first win and how many for the second, and then share the money according to this proportion.”
Fermat’s Method of Combinations

The game is like playing with a two-faced die and that the players roll four of them. Let \( a \) favor the first and \( b \) the second player. Thus all the faces with two \( a \)’s make the first player win; there are 11 of them. All the faces with three \( b \)’s make the second player win; there are 5 of them, so the total should be shared as 11:5.

“I have to tell you that this division for two players, based on combinations, is quite correct and good; but, if there are more than two players, it will not always be right, and I will tell you the reason for this difference.”
Gilles de Roberval’s Objection

“M. Roberval made me this objection: that it is wrong to make the division assuming that four rounds will be played, seeing that … one might play only two or three. … And thus he did not see why one presumed to make a correct division based on a false condition, that four rounds would be played. … The natural mode of play is that it stops after one of the players has won. At the least, if the result isn’t true, it hasn’t been demonstrated. He suspects we are being illogical.”

[Pascal answers Roberval by saying he has a “universal” and obvious method that produces the same result. But Pascal also attempts to persuade him that the “natural” course of play and the hypothetical play of all four rounds are “equal.”]
Pascal’s Analysis

Let’s pursue the same idea for three players needing (1,2,2). A decision will have been made within three rounds. The number of combinations of three dice with three faces is $3^3 = 27$.

There are 19 positions with an $a$, 7 with two $b$’s, and 7 with two $c$’s. But “if from that we conclude it would be necessary to give to each in the proportion 19:7:7, we would be hugely deceived, because there are several faces favorable” to two players at once.
“If the roll acc is made, the first and the third (players) will have the same right to the total … and therefore would divide the money in half, but if the roll aab is made, the first wins all.”

Pascal computes,

- **13** rolls give all to A, totaling $13 \times 1 = 13$.
- **6** rolls give half to A, totaling $6 \times \frac{1}{2} = 3$.
- **8** rolls give nothing to A, totaling **0**.

Out of $13 + 6 + 8 = 27$ rolls, A receives $13 + 3 + 0 = 16$, so he should get $16/27$ of the prize.

He checks by similarly computing $5 \frac{1}{2}$ each for B and C.
Pascal’s Objection

“That, it seems to me, is how the division should be made with combinations following your method, provided you might not have something else on this subject which I cannot know. But unless I deceive myself, this division is unfair.”

“The reason is that one is assuming something false, which is that three rounds will always be played, instead of the natural condition which is that one plays only until one of the players has achieved the number of rounds he lacks, in which case the game ends.” …

[The correct result is that] “to the first will belong 17, 5 to the second, and 5 to the third, out of 27.”
Was Fermat Wrong?

“I believe I have let you know here that the method of combinations is accidentally good for two players, as it also sometimes is among three players, such as [(1,1,2)]: because in these cases the number of rounds in which the game will be over does not allow two to win; but this is not general, and is not generally good except only when one is constrained to play exactly a given number of rounds. As a result, because you did not know my method when you had proposed to me the division of several players, but knew only that of combinations, I fear we might have different feelings about this subject. I beg of you to tell me how your research proceeds on this division, …”
Fermat’s Reply

September 25, 1654

I only find 17 combinations for the first [player] and 5 for each of the two others; for when you say that the combination ACC is good for the first and for the third, it seems that you no longer remember that everything done after one of the players has won, is good for nothing. …

That fiction of prolonging the game to a certain number of rounds serves only to help with the working-out, and (following my intuition) to make all chances equal. … If in place of three rounds you prolong the artificial game up to four, there will be not 27 combinations, but 81, and it would be necessary to see how many combinations will make the first win a round before either of the others wins two … [there will be] 51.
Fermat Answers Roberval

Key:
Arrows denote outcomes.
A, B, C show in whose favor.
Colored dots indicate where the game is won (and the color indicates the winner).
Numbers (1/3, etc.) are the chances of reaching each leaf.
The End

Pascal to Fermat, October 27 1654.

I admire your division method, much more because I understand it right well; it is entirely yours, and has nothing in common with mine, and easily arrives at the same goal.

(Pascal abandoned mathematics on November 23.)

Fermat to Carcavi, August 9 1659.

I value [Pascal’s] genius and I believe him quite capable of achieving whatever he will set out to do. ... I will briefly send M. Pascal all my principles and first proofs, and I tell you in advance that he will derive things not only new and heretofore unknown, but also surprising.